On the partition of numbers *

Leonhard Euler

§1 The Problem *about the Partition of Numbers* was first propounded to me by Professor Naude; he asked, in how many different ways a given integer (for, here we always consider only integer and positive numbers) can be an aggregate of two or three or four or in general of arbitrarily many numbers. Or, what reduces to the same, in how many different ways a given number can be split into either two or three or four or arbitrarily many parts, whence this problem very appropriately was given the name of *Partition of Numbers*. But the Professor's problem is usually propounded in two parts: First, it demands only the ways of partitions, in which the single parts, into which the propounded number is to be resolved, are different; but then having omitted this condition of inequality, it requires all possible ways of partitions, no matter whether certain parts are equal to each other or all are different. But it is perspicuous that in this last case the number of partitions mostly is a lot greater than in the first, since not only all partitions, which satisfy in the first case, at the same time resolve the latter, but also mostly many others containing equal parts are added.

§2 In order to see the complexity of this problem more clearly, I will mention some simpler cases, which can actually be handled easily by actual enumeration of the partitions. If it is in question, in how many different ways the number 6 can be resolved into two parts, it is immediately clear that this can be done in three ways, since

*Original title: „De partitione numerorum ", first published in „*Novi Commentarii academiae scientiarum Petropolitanae* 3, 1753, pp. 125-169", reprinted in „*Opera Omnia*: Series 1, Volume 2, pp. 254 - 294", Eneström-Number E191, translated by: Alexander Aycock for the „Euler-Kreis Mainz"

$$6 = 1 + 5 = 2 + 4 = 3 + 3,$$

if the parts are not necessarily different, of course. But if only different parts are desired, the last partition $3 + 3$ is to be omitted and in this case the number 6 can be split into two different parts in only two ways. If an odd number, as 9, is propounded to be split into two parts, four partitions will arise, which are

$$9 = 1 + 8 = 2 + 7 = 3 + 6 = 4 + 5;$$

since there no equal parts occur, the number 9 can be split into two parts in four ways, no matter whether equal parts are excluded or not. If more than two parts are desired, as if it is asked, in how many ways the number 12 can be split into three parts, one will be able to do this in the following 12 ways:

$$\begin{array}{lll}
12 = 1 + 1 + 10, & 12 = 1 + 2 + 9, & 12 = 1 + 3 + 8, \\
12 = 1 + 4 + 7, & 12 = 1 + 5 + 6, & 12 = 2 + 2 + 8, \\
12 = 2 + 3 + 7, & 12 = 2 + 4 + 6, & 12 = 2 + 5 + 5, \\
12 = 3 + 3 + 6, & 12 = 3 + 4 + 5, & 12 = 4 + 4 + 4.
\end{array}$$

But if the equal parts are excluded, one will have to answer that the number 12 can be split into three parts in only 7 ways.

§3 Hence it is easily seen, if the number to be split was three or four times greater than the number of parts, into which it must be resolved, that then the number of partitions becomes so large, that by an actual enumeration it can be obtained only with a lot of work. And in this task one must also not trust induction, which, as it will become clear to anyone trying it, is mostly false, if one wants to make conclusions for higher composited cases from the enumerations done for the simpler cases. So it will become clear from the method to be explained later that the number 50 can be split into seven parts in 8946 ways, not having excluded the equality of the parts; but if equal parts are excluded, only 522 partitions will remain. Further, the number 42 can be resolved into 20 parts in thousand different ways. But if it is in question, in how many different ways the number 125 can be split into 12 parts, which are all different to each other, one will find that this can happen in 64707 ways.

§4 As here all the integer numbers can take the place of the parts, so this problem can be varied to infinity, depending on whether the numbers constituting the parts are restricted or not. So it will be another problem, if it is in question, in how many different ways the given number n can be resolved into p parts, of which none exceeds a given number m. The number of parts can also be omitted, as if it is asked, in how many different ways the number 6 can be produced from these numbers 1, 2, 3, 4 by addition, which can be done the following 9 ways:

$$
\begin{array}{ll}
6 = 1+1+1+1+1+1 & 6 = 1+1+1+3 \\
6 = 1+1+1+1+2 & 6 = 1+1+4 \\
6 = 1+1+2+2 & 6 = 1+2+3 \\
6 = 2+2+2 & 6 = 2+4 \\
 & 6 = 3+3
\end{array}
$$

Or even the kind of the numbers, which constitute the parts, can be prescribed; one can require the parts to be either odd numbers or squares or triangular numbers or of any other kind. So if it is in question, in how many different ways a given number can be the sum of four squares, this question will extend to this kind of problem. So a long time ago also the partition of numbers into parts, which are terms of this geometric progression 1, 2, 4, 8, 16, 32 etc., was considered and every number was observed that it can be composed in only one single way from these numbers 1, 2, 4, 8, 16, 32 etc. by addition. After Stiefel this question was asked by van Schooten in his *Exercitationes*, where he showed that weights of 1, 2, 4, 8, 16, 32 etc. mass units suffice to weigh any arbitrary mass. And to show this he used induction. Therefore, it will be convenient to have demonstrated the truth of this claim rigorously.

§5 Therefore, I will explain a certain and safe method, how this and other similar problems must be resolved, so that induction, which is usually applied for the solution of problems of this kind, is not necessary at all. For this, I use the following very well-known lemma:

If this product

$$(1+az)(1+bz)(1+cz)(1+dz)(1+ez) \text{ etc.},$$

no matter whether the number of factors is finite or infinite, is expanded by actual multiplication, that a form of this kind results

$$1 + Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + \text{etc.},$$

the coefficient of the second term A will be the sum of all quantities a, b, c, d, e etc. The coefficient B on the other hand will be the sum of products of two different quantities. The coefficient C will be the sum of products of three different quantities; and the coefficient D will be the sum of products of four of these same quantities, and so forth.

For, the same quantity, say a, or any other can never be contained more than once in products of this kind. Hence this lemma provides me with the foundation of partition into different parts.

§6 But if the equality of the parts is not excluded, I use this lemma:

If this form

$$\frac{1}{(1-az)(1-bz)(1-cz)(1-dz)(1-ez)\text{etc.}},$$

no matter whether the number of the factors constituting the denominator is finite or infinite, after the expansion of the denominator by actual multiplication, is expanded into a series of the following form by means of division

$$1 + Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + \text{etc.},$$

then A will be, as before, the sum of the quantities $a + b + c + d + e + etc.$. But the coefficient B will be the sum of products of two not necessarily different of these quantities; of course, it will be

$$B = aa + ab + bb + ac + bc + cc + ad + bd + cd + dd + ae + \text{etc.}$$

In like manner the coefficient C will be the sum of products of three not necessarily different of these quantities a, b, c, d, e etc. And the coefficient D will be the sum of products of four not necessarily different of these quantities and so forth.

And hence this lemma will open the way to find partitions, in which the equality of the parts is not excluded.

§7 But since in the propounded problem the question is not about products but about the sum of numbers, I substitute the powers x^p, x^q, x^r, x^s, x^t etc. for the quantities a, b, c, d, e etc. For, so in the products of two quantities of such a kind those powers will occur, whose exponents are the sum of two numbers from the series p, q, r, s, t etc. In like manner, the products of three consist of powers of such a kind, whose exponents are the sums of three numbers of the same series p, q, r, s etc. And the products of four will be the powers, whose exponents are the aggregates of four of these numbers, and so forth. And so, what was mentioned before on products, is now transferred to sums and certainly in such a way that, if the first lemma is applied, the sums are conflated of only different parts, but if the second lemma is used, the equality of the parts is not excluded. Therefore, both lemmas must be used this way for the solution of these questions raised before.

§8 Therefore, let us solve this first problem:

To find, in how many different ways the given number N can be split into p different parts.

Since for this all positive integer numbers are suitable to constitute the parts, for the series of the above exponents the series of the natural numbers 1, 2, 3, 4, 5, 6 etc. is to be taken. Therefore, according to the second lemma form this expression

$$s = (1+xz)(1+x^2z)(1+x^3z)(1+x^4z)(1+x^5z)\text{etc. to infinity},$$

which, having actually done the multiplication, is expanded into this series

$$s = 1 + Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + \text{etc.},$$

and it will be

$$A = x^1 + x^2 + x^3 + x^4 + x^5 + x^6 + \text{etc.},$$

which is the aggregate of all powers of x. Further, since B is the sum of products of two different terms of the series A, B will be the sum of all powers of x, whose exponents are aggregates of two different numbers; and since the same power can result more than once, it will have a numerical coefficient indicating, in how many ways the power is a product of two terms of the series A or in how many different ways its exponent can be the sum of two different numbers. But, actually multiplying two terms of the series A, one will find

$$B = x^3 + x^4 + 2x^5 + 2x^6 + 3x^7 + 3x^8 + 4x^9 + 4x^{10} + \text{etc.}$$

Every coefficient of this series indicates, in how many different ways the exponent of the respective power of x can be split into two different parts. Therefore, having continued this series to infinity by means of the law to be found later, the case of the problem requiring the partition into two parts is now solved.

§9 Further, the quantity C, since it contains all products resulting by multiplying three different terms of the series A, will consist of the series of powers of x, whose exponents are the sums of three different numbers. And the same power will occur so often in that series C as its exponent can result from three different numbers by addition, and one will find

$$C = x^6 + x^7 + 2x^8 + 3x^9 + 4x^{10} + 5x^{11} + 7x^{12} + 8x^{13} + 10x^{14} + \text{etc.}$$

Any arbitrary coefficient of this series indicates, in how many different ways the exponent of a power of x can be split into three different parts; and so it follows from the term $8x^{13}$ that the number 13 can be split into three different parts in eight different ways, which are

$$\begin{array}{ll} 13 = 1 + 2 + 10, & 13 = 2 + 3 + 8, \\ 13 = 1 + 3 + 9, & 13 = 2 + 4 + 7, \\ 13 = 1 + 4 + 8, & 13 = 2 + 5 + 6, \\ 13 = 1 + 5 + 7, & 13 = 3 + 4 + 6. \end{array}$$

Therefore, this series C, if continued to infinity, will serve for splitting all numbers into three different parts.

§10 Further, the quantity D, since it contains all products of four different terms of the series $A = x^1 + x^2 + x^3 + x^4 + \text{etc.}$, will consist of the series of those powers of x, whose exponents are the aggregate of four different numbers; and in this series any arbitrary power will have a coefficient indicating, in how many different ways its exponent can result from addition of four different numbers. But one will find

$$D = x^{10} + x^{11} + 2x^{12} + 3x^{13} + 5x^{14} + 6x^{15} + 9x^{16} + 11x^{17} + \text{etc.}$$

Therefore, this series, if continued to infinity, will show, in how many different ways each number can be the sum of four different parts. For example, from the term $9x^{16}$ it is seen that the number 16 can be split into different parts in nine ways.

§11 If we proceed further this way, it will become plain that the letter E will be a power series in x of such a nature that the coefficient of each term indicates, in how many different ways the exponent of x can be split into five different parts. But it will be

$$E = x^{15} + x^{16} + 2x^{17} + 3x^{18} + 5x^{19} + 7x^{20} + 10x^{21} + 13x^{22} + \text{etc.}$$

In like manner, the value of the letter F will be the series serving for the partitions into six different parts and the letters G, H, I etc. for the partitions into seven, eight, nine etc. parts and they will be

$$\begin{aligned} F &= x^{21} + x^{22} + 2x^{23} + 3x^{24} + 5x^{25} + 7x^{26} + 11x^{27} + 14x^{28} + \text{etc.}, \\ G &= x^{28} + x^{29} + 2x^{30} + 3x^{31} + 5x^{32} + 7x^{33} + 11x^{34} + 15x^{35} + \text{etc.} \\ &\text{etc.} \end{aligned}$$

Hence it is understood that the exponent of the first term of each series is the triangular number of the number of propounded parts, but then the coefficient so of this as of the second term is $= 1$. The reason for this is easily seen; for, the smallest number, which is the number of seven different parts, obviously is $= 1 + 2 + 3 + 4 + 5 + 6 + 7 = \frac{1}{2}7 \cdot 8 =$ the triangular number of seven, and this number as the following greater by one unit can not be split into seven different parts in more than one way.

§12 Therefore, the whole task reduces to the convenient formation of the series B, C, D, E, F etc., that not that itself, what it is in question, of course the number of partitions, is used for the formation of each series. And first, certainly the law of the progressions A and B is clear, since the coefficients of the first are all 1, the coefficients of the second series on the other hand are the terms of the series of the natural numbers, where each number appears twice; but the law of the following series on the other hand is less obvious, and they were actually continued so far as seen here, by using the method to be explained later to find the number of the partitions of the exponents. Therefore, the values of these letters A, B, C, D etc. must be investigated in another way, whence this problem arises:

To find the values of the letters A, B, C, D, E etc. such that the sum of this series

$$s = 1 + Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + \text{etc.}$$

becomes equal to this expression

$$s = (1 + xz)(1 + x^2z)(1 + x^3z)(1 + x^4z)(1 + x^5z)\text{etc.}$$

Therefore, for this aim, the relation among these expressions, and how the one must be changed to transform into the other, must be considered.

§13 Since the value s of each of both expressions is the same, their relation to each other will remain the same, if in both of them an arbitrary other quantity is written instead of z. Therefore, let us put xz instead of z in both of them and let us call the value resulting in both expressions t and first it will be

$$t = 1 + Axz + Bx^2z^2 + Cx^3z^3 + Dx^4z^4 + \text{etc.};$$

but then the other expression will be transformed into this one

$$t = (1 + x^2z)(1 + x^3z)(1 + x^4z)(1 + x^5z)\text{etc.};$$

if this last value of t is compared to the second value of s, which was

$$s = (1 + xz)(1 + x^2z)(1 + x^3z)(1 + x^4z)\text{etc.},$$

it will soon become plain that $s = (1 + xz)t$. Since this relation must also hold in the other values of s and t, this will give us these equations

$$\begin{aligned} s &= 1 + A\ \ z + B\ \ \ z^2 + C\ \ \ z^3 + D\ \ \ z^4 + \text{etc.},\\ (1+xz)t &= 1 + Axz + Bx^2z^2 + Cx^3z^3 + Dx^4z^4 + \text{etc.},\\ &\qquad\ + \ \ xz + Ax^2z^2 + Bx^3z^3 + Cx^4z^4 + \text{etc.} \end{aligned}$$

Hence by equating the homogeneous terms to each other it will be

$$
\begin{aligned}
A &= \frac{x}{1-x},\\
B &= \frac{Ax^2}{1-x^2} = \frac{x^3}{(1-x)(1-x^2)},\\
C &= \frac{Bx^3}{1-x^3} = \frac{x^6}{(1-x)(1-x^2)(1-x^3)},\\
D &= \frac{Dx^4}{1-x^4} = \frac{x^{10}}{(1-x)(1-x^2)(1-x^3)(1-x^4)},\\
E &= \frac{Ex^5}{1-x^5} = \frac{x^{15}}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)}\\
&\text{etc.}
\end{aligned}
$$

§14 Therefore, the series, which were observed above to arise for the letters A, B, C, D, E etc., result from the expansion of the fractions we found here, whence it is known that the series A is a geometric series, namely $A = x + x^2 + x^3 + x^4 + x^5 + \text{etc.}$, which, what is certainly very clear, indicates that each number can consist of one single integer number in one single way. The remaining series B, C, D. E etc. on the other hand are recurring series, whose relation scale for each fraction will become plain from the denominator, if it is expanded by multiplication. To show this, let us neglect the numerators for the moment; they are those powers of x, whose exponents are the triangular numbers, and instead of them let us write 1. Therefore, let

$$
\begin{aligned}
\frac{A}{x} &= 1 + \alpha' x + \beta' x^2 + \gamma' x^3 + \delta' x^4 + \varepsilon' x^5 + \cdots + v' x^n + \cdots = \mathfrak{A},\\
\frac{B}{x^3} &= 1 + \alpha'' x + \beta'' x^2 + \gamma'' x^3 + \delta'' x^4 + \varepsilon'' x^5 + \cdots + v'' x^n + \cdots = \mathfrak{B},\\
\frac{C}{x^6} &= 1 + \alpha''' x + \beta''' x^2 + \gamma''' x^3 + \delta''' x^4 + \varepsilon''' x^5 + \cdots + v''' x^n + \cdots = \mathfrak{C},\\
\frac{D}{x^{10}} &= 1 + \alpha^{\mathrm{IV}} x + \beta^{\mathrm{IV}} x^2 + \gamma^{\mathrm{IV}} x^3 + \delta^{\mathrm{IV}} x^4 + \varepsilon^{\mathrm{IV}} x^5 + \cdots + v^{\mathrm{IV}} x^n + \cdots = \mathfrak{D},\\
\frac{E}{x^{15}} &= 1 + \alpha^{\mathrm{V}} x + \beta^{\mathrm{V}} x^2 + \gamma^{\mathrm{V}} x^3 + \delta^{\mathrm{V}} x^4 + \varepsilon^{\mathrm{V}} x^5 + \cdots + v^{\mathrm{V}} x^n + \cdots = \mathfrak{E},\\
\frac{F}{x^{21}} &= 1 + \alpha^{\mathrm{VI}} x + \beta^{\mathrm{VI}} x^2 + \gamma^{\mathrm{VI}} x^3 + \delta^{\mathrm{VI}} x^4 + \varepsilon^{\mathrm{VI}} x^5 + \cdots + v^{\mathrm{VI}} x^n + \cdots = \mathfrak{F}
\end{aligned}
$$

etc.

§15 Therefore, the solution of the question is reduced to finding the series $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$ etc., which are obviously recurring. And certainly the first series $\mathfrak{A}$, since $\mathfrak{A} = \frac{1}{1-x}$, is even a geometric series, and $\alpha' = 1$, $\beta' = 1$, $\gamma' = 1$, $\delta' = 1$ etc., which is perspicuous per se. But the series $\mathfrak{B}$, since

$$\mathfrak{B} = \frac{1}{(1-x)(1-x^2)} = \frac{1}{1-x-x^2+x^3},$$

will be recurring while the relation scale is $+1$, -1, $+1$; hence it will be

$$\begin{aligned}
\alpha'' &= 1, \\
\beta'' &= \alpha'' + 1, \\
\gamma'' &= \beta'' + \alpha'' - 1, \\
\delta'' &= \gamma'' + \beta \ \ - \alpha'', \\
\varepsilon'' &= \delta'' + \gamma'' - \beta'', \\
\zeta'' &= \varepsilon'' + \delta'' - \gamma''
\end{aligned}$$

etc.

In like manner, the series $\mathfrak{C}$, because of

$$\mathfrak{C} = \frac{1}{(1-x)(1-x^2)(1-x^3)} = \frac{1}{1-x-x^2+x^4+x^5-x^6},$$

will be recurring and will have the relation scale $+1$, $+1$, 0, -1, -1, $+1$. Hence it will be

$$\begin{aligned}
\alpha''' &= 1, \\
\beta''' &= \alpha''' + 1, \\
\gamma''' &= \beta''' + \alpha''' + *, \\
\delta''' &= \gamma''' + \beta''' + * - 1, \\
\varepsilon''' &= \delta''' + \gamma''' + * - \alpha''' - 1, \\
\zeta''' &= \varepsilon''' + \delta''' + * - \beta''' - \alpha''' + 1, \\
\eta''' &= \zeta''' + \varepsilon''' + * - \gamma''' - \beta''' + \alpha''', \\
\theta''' &= \eta''' + \zeta''' + * - \delta''' - \gamma''' + \beta'''
\end{aligned}$$

etc.

In like manner, the following series will be seen to be recurring and one will be able to assign their relation scale this way. But even though this way these series can be formed easily, I will nevertheless, having put aside this direct way, exhibit a much more convenient way to form each of these series from the preceding one, after I will have mentioned an observation of greatest importance.

§16 Because $\mathfrak{B} = \frac{1}{(1-x)(1-x^2)}$, having expanded $\mathfrak{B}$ into a series, it is plain that each power of x must occur as often as it can arise from the powers x^1, x^2 by multiplication or as often as its exponent can be produced from the numbers 1 and 2 by addition. So, because

$$\mathfrak{B} = 1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 + \cdots + v''x^n + \cdots,$$

from the term $3x^4$ it is understood that the number 4 can result from the numbers 1 and 2 by addition in three ways, which are

$$4 = 1+1+1+1, \quad 4 = 1+1+2 \quad \text{and} \quad 4 = 2+2.$$

Therefore, by considering the term $v''x^n$ the coefficient v'' will indicate, in how many ways the exponent n can be produced from the numbers 1 and 2 by addition. Therefore, because $B = \mathfrak{B}x^3$, in the series B one will have the term $v''x^{n+3}$; since this indicates that the number $n+3$ can be split into two different parts in so many ways as the coefficient v'' contains units it is obvious that the number $n+3$ can be split into two parts in so many ways as the number n can be produced by addition from the numbers 1 and 2.

§17 Further, since $\mathfrak{C} = \frac{1}{(1-x)(1-x^2)(1-x^3)}$, it is plain that in this series $\mathfrak{C}$ each power of x must occur so often as it can arise form the powers x^1, x^2, x^3 by multiplication or, what is the same, as its exponent can be produced from the numbers 1, 2, 3 by addition. So, because

$$\mathfrak{C} = 1 + x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + 7x^6 + \cdots + v'''x^n + \cdots,$$

from whose term $5x^5$ it will be seen that the exponent 5 can be produced from the numbers 1, 2, 3 by addition in five ways, which are

$$5 = 1+1+1+1+1, \quad 5 = 1+1+1+2, \quad 5 = 1+1+3,$$

$$5 = 1 + 2 + 2, \quad 5 = 2 + 3.$$

But considering the term $v'''x^n$ in general the coefficient v''' will indicate, in how many different ways the number n can result from the numbers 1, 2, 3 by addition. Therefore, since $C = \mathfrak{C}x^6$, one will have the terms $v'''x^{n+6}$ in the series C; this term indicates that the number $n+6$ can be split into three different parts in so many ways as the coefficient v''' contains units. Hence it follows that the number $n+6$ can be split into three different parts in as many ways as the number n can result from the numbers 1, 2, 3 by addition.

§18 It is not necessary that we prosecute this any longer, since hence it is already sufficiently clear that the number $n+10$ can be split into four different parts in so many different ways as the number n can result from the four numbers 1, 2, 3, 4 by addition. In like manner the number $n+15$ can be split into five different parts in so many ways as the number n can result from the five numbers 1, 2, 3, 4, 5 by addition. Therefore, generally the number $n + \frac{m(m+1)}{2}$ can be split into m different parts in so many different ways as the number n can result from the numbers 1, 2, 3, 4, $\cdots$ m by addition. Therefore, if it is in question, in how many different ways the number N can be split into m different parts, the answer will be found, if the number of cases is investigated in which the number $N - \frac{m(m+1)}{2}$ can result from the numbers 1, 2, 3, 4, $\cdots$ m by addition.

§19 Therefore, this way the resolution of the propounded question on the partition of an arbitrary number into arbitrary different parts is reduced to the solution of another problem already addressed above in which it is in question in how many different ways an arbitrary number can result from some of the terms of the arithmetic progression 1, 2, 3, 4, 5 etc. by addition. To explain this more clearly, let us introduce a more convenient notation. Therefore, let:

$n^{(2)}$ denote the number of cases in which the number n can be formed from the numbers 1, 2 by addition;

let $n^{(3)}$ denote the number of cases in which the number n can be formed from the numbers 1, 2, 3 by addition;

and let $n^{(m)}$ denote the number of cases in which the number n can be formed from the numbers 1, 2, 3, $\cdots$ m by addition.

Therefore, having introduced the values of characters of this kind, the propounded problem will be resolved this way. If it is in question, in how many different ways the number N can be split into m different parts, the number of cases in question will be expressed by the character $\left(N - \frac{m(m+1)}{1 \cdot 2}\right)^{(m)}$ indicating, in how many different ways the number $N - \frac{m(m+1)}{2}$ can result from the numbers 1, 2, 3, $\cdots$ m by addition.

§20 Also the solution of the other problem propounded by Naude is reduced to this same question, wherefore it will be convenient to resolve this problem before we attempt a further expansion of the characters we just introduced; for, we will solve three problems, which seem to be rather different in their nature, at once. The problem is the following:

To find, in how many different way a given number N can by split into p not necessarily different parts.

Since here the equality of the parts is not excluded, I will contemplate the following form containing the solution of this question

$$s = \frac{1}{(1-xz)(1-x^2z)(1-x^3z)(1-x^4z)(1-x^5z)\text{etc.}},$$

which, having expanded into a power series in z, yields this series

$$s = 1 + Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + \text{etc.},$$

and, as we noted above (§ 6), the coefficient A will be the sum of all terms of this series x, x^2, x^3, x^4, x^5 etc. or $A = x^1 + x^2 + x^2 + x^3 + x^4 + x^5 + x^6 + \text{etc.}$ which is the same series we obtained in the solution of the preceding problem for the letter A.

§21 Further, B is the sum of the product of two not necessarily different terms of the series A. Hence B will be the sum of all powers of x whose exponents are the aggregates of two either equal or different numbers; and since the same power can result more than once this way, it will have a numerical coefficient indicating, in how many ways the power is the product of two terms of the series A or in how many different ways its exponent can be the sum of two either different or equal numbers. From this one will find

$$B = x^2 + x^3 + 2x^4 + 2x^5 + 3x^6 + 3x^7 + 4x^8 + 4x^9 + \text{etc.},$$

any coefficient of which series indicates, in how many different ways the exponent of the corresponding power of x can be split into two parts. Therefore, having continued this series to infinity, the case in which the partition into two parts is required is easily solved.

§22 Further, the quantity C, since it contains all products resulting from the multiplication three either different or equal terms of the series A, will consist of a series of powers of x whose exponents are the sums of three positive integer numbers. And the same power x^n will occur so often in the series C as its exponent n can result from three either equal or different numbers by addition. But it will be

$$C = x^3 + x^4 + 2x^5 + 3x^6 + 4x^7 + 5x^8 + 7x^9 + 8x^{10} + 10x^{11} + \text{etc.},$$

any coefficient of each series indicates, in how many different ways the exponent of the corresponding power of x can be split into three either or different parts. So it is concluded from the term $8x^{10}$ that the number 10 can be split into three parts in eight ways; these partitions are

$$\begin{aligned} 10 &= 1 + 1 + 8, \qquad & 10 &= 2 + 2 + 6, \\ 10 &= 1 + 2 + 7, \qquad & 10 &= 2 + 3 + 5, \\ 10 &= 1 + 3 + 6, \qquad & 10 &= 2 + 4 + 4, \\ 10 &= 1 + 4 + 5, \qquad & 10 &= 3 + 3 + 4. \end{aligned}$$

Therefore, this series C, if it is continued to infinity, will serve for partitioning all numbers into three parts.

§23 In like manner, the quantity D, since it contains all products of four not necessarily different terms of the series $A = x + x^2 + x^3 + x^4 + \text{etc.}$, will consist of a series of powers of x, whose exponents are the aggregates of four either equal or different numbers. Therefore, in this series any arbitrary power of x will have a coefficient of such a kind which indicates, in how many different ways its exponent can result by the addition of four numbers. But hence one will find

$$D = x^4 + x^5 + 2x^6 + 3x^7 + 5x^8 + 6x^9 + 9x^{10} + 11x^{11} + \text{etc.}$$

Therefore, this series, if continued to infinity, will show, in how many different ways any arbitrary number can be split into four parts. So it is concluded from the term $9x^{10}$ that the number 10 can be split into four parts in nine ways; these partitions are

$$\begin{aligned} 10 &= 1+1+1+7, & 10 &= 1+2+2+5, \\ 10 &= 1+1+2+6, & 10 &= 1+2+3+4, \\ 10 &= 1+1+3+5, & 10 &= 1+3+3+3, \\ 10 &= 1+1+4+4, & 10 &= 2+2+2+4, \end{aligned}$$
$$10 = 2+2+3+3.$$

§24 Proceeding further this way it will become clear that the letter E will be a series of such a kind of the powers of x that the coefficient of each term indicates, in how many different ways the exponent of x can be split into 5 parts. But it will be

$$E = x^5 + x^6 + 2x^7 + 3x^8 + 5x^9 + 7x^{10} + 10x^{11} + 13x^{12} + \text{etc.}$$

In the same way the value of the letter F will be a series revealing the partitions into six parts and the value of the letters G, H, I will reveal the partitions into seven, eight, nine etc. parts; but it will be

$$\begin{aligned} F &= x^6 + x^7 + 2x^8 + 3x^9 + 5x^{10} + 7x^{11} + 11x^{12} + 14x^{13} + \text{etc.}, \\ G &= x^7 + x^8 + 2x^9 + 3x^{10} + 5x^{11} + 7x^{12} + 11x^{13} + 15x^{14} + \text{etc.} \end{aligned}$$

If these series are compared to those we found in the solution of the above problem for the same letters, it will soon become clear that the whole difference consists only of the powers of x and the coefficients proceed similarly in both series. But to not rely on induction here, let us demonstrate this agreement as follows.

§25 Let us, as above, consider two values of s which are

$$s = 1 + Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + \text{etc.}$$
$$s = \frac{1}{(1-xz)(1-x^2z)(1-x^3z)(1-x^4z)(1-x^5z)\text{etc.}};$$

let these, if one puts xz instead of z everywhere, go over into t and it will be

$$t = 1 + Axz + Bx^2z^2 + Cx^3z^3 + Dx^4z^4 + Ex^5z^5 + \text{etc.}$$
$$t = \frac{1}{(1-x^2z)(1-x^3z)(1-x^4z)(1-x^5z)\text{etc.}}.$$

Hence, if the second values of s and t are compared to each other, it will soon become evident that $s = \frac{t}{1-xz}$ or $t = (1-xz)s$; since this same relation also has to hold between the first value of letters s and t, it will be

$$\begin{aligned} t &= 1 + Axz + Bx^2z^2 + Cx^3z^3 + Dx^4z^4 + Ex^5z^5 + \text{etc.}, \\ (1-xz)s &= 1 + \quad Az + \quad Bz^2 + \quad Cz^3 + \quad Dz^4 + \quad Ez^5 + \text{etc.} \\ &\quad - \quad xz - Axz^2 - Bxz^3 - Cxz^4 - Dxz^5 - \text{etc.} \end{aligned}$$

Hence by equating the coefficients one finds

$$\begin{aligned} A &= \frac{x}{1-x}, \\ B &= \frac{Ax}{1-xx} = \frac{x^2}{(1-x)(1-x^2)}, \\ C &= \frac{Bx}{1-x^3} = \frac{x^3}{(1-x)(1-x^2)(1-x^3)}, \\ D &= \frac{Cx}{1-x^4} = \frac{x^4}{(1-x)(1-x^2)(1-x^3)(1-x^4)} \\ &\text{etc.} \end{aligned}$$

§26 From these formulas it is understood that these series are not only also recurring as the ones above but also the rule for the coefficients is the same. Hence, if having neglected the numerators one puts that

$$\mathfrak{A} = \frac{1}{1-x}, \qquad A = \mathfrak{A}x,$$

$$\mathfrak{B} = \frac{1}{(1-x)(1-x^2)}, \qquad B = \mathfrak{B}x^2,$$

$$\mathfrak{C} = \frac{1}{(1-x)(1-x^2)(1-x^3)}, \qquad C = \mathfrak{C}x^3,$$

$$\mathfrak{D} = \frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)}, \qquad D = \mathfrak{D}x^4$$

etc., etc.,

the partition of any number into an arbitrary amount of either equal or different parts depends on the formation of the series $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. which, as we observed before, indicate, in how many different ways a certain number can be produced from several initial terms of the series 1, 2, 3, 4, 5 etc. by addition. Therefore, since $B = \mathfrak{B}x^2$, the number $n+2$ can be split into two parts in so many ways as the number n can be produced from the numbers 1 and 2 by addition. In like manner, since $C = \mathfrak{C}x^3$, the number $n+3$ will be split into three parts in so many ways as the number n can be composed of the numbers 1, 2, 3 by addition. And generally the number $n+m$ can be split into m either equal or different parts in so many different ways as the number n can be produced from the numbers 1, 2, 3, $\cdots$ m by addition.

§27 Therefore, even this problem will depend on the solution of the question, in how many different ways a given number can result from several initial terms of the series 1, 2, 3, 4. Therefore, if as above [§ 19] this symbol $n^{(m)}$ denotes the number of ways, in which the number n can be composed of the numbers 1, 2, 3, $\cdots$ m by addition or in which the number n can be split into arbitrarily many parts, none of which is larger than the number m, even this problem can be solved by characters of this kind. Of course, $n^{(m)}$ will indicate, in how many different ways the number $n+m$ can be split into m either equal or different parts. Hence, if it is question, in how many ways the number N can be split into m either equal or different parts, the number of of ways in question will be indicated by this formula $(N-m)^{(m)}$. Therefore, if this problem is compared to the preceding one, it will be perspicuous that the number $n+m$ can be split into m either equal or different parts in as many ways as the number $n+\frac{m(m+1)}{2}$ can be split into m parts.

§28 Therefore, the solution of both problems propounded by Naude are reduced to the task to define, in how many different ways any arbitrary number n can be formed from the numbers $1, 2, 3, \cdots m$ by addition, or to the investigation of the value of the character $n^{(m)}$. Therefore, let us see, how this new problem can be resolved most conveniently from the formulas found before. And at first, if $m=1$, since any arbitrary number can be found in one single way from unities only by addition, it will be

$$n^{(1)}=1,$$

what the first formula $\mathfrak{A}=\frac{1}{1-x}$ or the series formed from this, namely

$$\mathfrak{A}=1+x+x^2+x^3+x^4+x^5+\text{etc.},$$

obviously indicates.

§29 Since the series $\mathfrak{B}=\frac{1}{(1-x)(1-x^2)}$ indicates, in how many ways any number can be formed from the numbers 1 and 2 by addition, in this series the coefficients of the power x^n will be $=n^{(2)}$; for, this expression was assumed to denote, in how many ways the number n can result from the number 1 and 2 by addition. Therefore, it will hence be

$$\mathfrak{B}=1+1^{(2)}x+2^{(2)}x^2+3^{(2)}x^3+4^{(2)}x^4+5^{(2)}x^5+6^{(2)}x^6+\text{etc.}$$

and likewise it will be

$$\mathfrak{A}=1+1^{(1)}x+2^{(1)}x^2+3^{(1)}x^3+4^{(1)}x^4+5^{(1)}x^5+6^{(1)}x^6+\text{etc.}$$

Further, since $\mathfrak{A}=\frac{1}{1-x}$ and $\mathfrak{B}=\frac{1}{(1-x)(1-x^2)}$, it will be $\mathfrak{A}=\mathfrak{B}(1-x^2)$, whence the following relation among these series results

$$\begin{array}{rcl}
\mathfrak{A} & = & 1+1^{(1)}x+2^{(1)}x^2+3^{(1)}x^3+4^{(1)}x^4+5^{(1)}x^5+6^{(1)}x^6+\text{etc.}, \\
\mathfrak{B} & & 1+1^{(2)}x+2^{(2)}x^2+3^{(2)}x^3+4^{(2)}x^4+5^{(2)}x^5+6^{(2)}x^6+\text{etc.} \\
-\mathfrak{B}x^2 & = & \qquad\quad -\quad x^2-1^{(2)}x^3-2^{(2)}x^4-3^{(2)}x^5-4^{(2)}x^6-\text{etc.}
\end{array}$$

Hence, if the corresponding terms are compared, it will be

$$
\begin{array}{lll}
1^{(2)} = 1^{(1)}, & 4^{(2)} = 4^{(1)} + 2^{(2)}, & 7^{(2)} = 7^{(1)} + 5^{(2)}, \\
2^{(2)} = 2^{(1)} + 1, & 5^{(2)} = 5^{(1)} + 3^{(2)}, & 8^{(2)} = 8^{(1)} + 6^{(2)}, \\
3^{(2)} = 3^{(1)} + 1^{(2)}, & 6^{(2)} = 6^{(1)} + 4^{(2)}, & 9^{(2)} = 9^{(1)} + 7^{(2)}.
\end{array}
$$

§30 Therefore, in general it will be

$$n^{(2)} = n^{(1)} + (n-2)^{(2)}.$$

Therefore, since $n^{(1)} = 1$, it will be $n^{(2)} = 1 + (n-2)^{(2)}$; and so the coefficients of the series $\mathfrak{B}$ will be determined in such a way that the last term is equal to the antepenultimate augmented by 1. Or since all coefficients of the series $\mathfrak{A}$ are 1, the series $\mathfrak{B}$ will be formed from the series $\mathfrak{A}$ in the following way:

$$
\begin{array}{rl}
\mathfrak{A} & = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + \text{etc.} \\
 & 1 \quad + 1 \quad + 2 \quad + 2 \quad + 3 \quad + 3 \quad + 4 \quad + 4 \quad + \text{etc.} \\
\hline
\mathfrak{B} & = 1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 + 4x^6 + 4x^7 + 5x^8 + 5x^9 + \text{etc.}
\end{array}
$$

Since the initial terms of the series $\mathfrak{B}$ are known, they are $1 + x$, of course, write them under the third and fourth term of the series $\mathfrak{A}$ and hence the third and the fourth term of the series $\mathfrak{B}$ will result, which further, if written under the fifth and sixth term of the series $\mathfrak{A}$ and added, will give the fifth and sixth term of the series $\mathfrak{B}$ and this way one can continue the series $\mathfrak{B}$ arbitrarily far. But it is plain that hence $n^{(2)} = \frac{1}{2}\left(n \begin{array}{c} +1 \\ +2 \end{array}\right)$; if n is an odd number, $n^{(2)} = \frac{1}{2}(n+1)$, but if n is an even number, it will be $n^{(2)} = \frac{1}{2}(n+2)$.

§31 Further, since $\mathfrak{C} = \frac{1}{(1-x)(1-x^2)(1-x^3)}$, it will be $\mathfrak{B} = \mathfrak{C}(1-x^3)$, whence, because the general term of the series $\mathfrak{C}$ is $n^{(3)}$, the following relation among $\mathfrak{B}$ and $\mathfrak{C}$ will result

$$
\begin{array}{rl}
\mathfrak{B} & = 1 + 1^{(2)}x + 2^{(2)}x^2 + 3^{(2)}x^3 + 4^{(2)}x^4 + 5^{(2)}x^5 + 6^{(2)}x^6 + \text{etc.}, \\
\mathfrak{C} & = 1 + 1^{(3)}x + 2^{(3)}x^2 + 3^{(3)}x^3 + 4^{(3)}x^4 + 5^{(3)}x^5 + 6^{(3)}x^6 + \text{etc.} \\
-\ \mathfrak{C}x^3 & \phantom{= 1 + 1^{(3)}x + 2^{(3)}x^2} -\ \ x^3 - 1^{(3)}x^4 - 2^{(3)}x^5 - 3^{(2)}x^6 + \text{etc.}
\end{array}
$$

If now here the coefficients of the two series are compared, it will be

$$\begin{array}{lll} 1^{(3)} = 1^{(2)}, & 4^{(3)} = 4^{(2)} + 1^{(3)}, & 7^{(3)} = 7^{(2)} + 4^{(3)}, \\ 2^{(3)} = 2^{(2)}, & 5^{(3)} = 5^{(2)} + 2^{(3)}, & 8^{(3)} = 8^{(2)} + 5^{(3)}, \\ 3^{(3)} = 3^{(2)} \ + 1 & 6^{(3)} = 6^{(2)} + 3^{(3)}, & 9^{(3)} = 9^{(2)} + 6^{(3)} \end{array}$$

and generally

$$n^{(3)} = n^{(2)} + (n-3)^{(3)}.$$

Therefore, the series $\mathfrak{C}$ is easily formed from the series $\mathfrak{B}$ and its preceding terms in the following way. But let us omit the powers of x, since we are only interested in the coefficients:

$$\begin{array}{rl} \mathfrak{B} = & 1 + 1 + 2 + 2 + 3 + 3 + 4 + 4 + \ \ 5 + \ \ 5 + \ \ 6 + \ \ 6 + \text{etc.} \\ & \qquad\qquad\quad\ 1 + 1 + 2 + 3 + 4 + \ \ 5 + \ \ 7 + \ \ 8 + 10 + \text{etc.} \\ \hline \mathfrak{C} = & 1 + 1 + 2 + 3 + 4 + 5 + 7 + 8 + 10 + 12 + 14 + 16 + \text{etc.} \end{array}$$

Of course, write the series $\mathfrak{C}$ under the series $\mathfrak{B}$, starting from the fourth term, and depending on how this way the series $\mathfrak{C}$ results by addition, so it will also be continued under the series $\mathfrak{B}$.

§32 Further, since $\mathfrak{D} = \frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)}$, it will be $\mathfrak{C} = \mathfrak{D}(1-x^4)$. Therefore, in the same way as before one will find

$$\begin{array}{lll} 1^{(4)} = 1^{(3)}, & 4^{(4)} = 4^{(3)} + 1, & 7^{(4)} = 7^{(3)} + 3^{(4)}, \\ 2^{(4)} = 2^{(3)}, & 5^{(4)} = 5^{(3)} + 1^{(4)}, & 8^{(4)} = 8^{(3)} + 4^{(4)}, \\ 3^{(4)} = 3^{(3)}, & 6^{(4)} = 6^{(3)} + 2^{4)}, & 9^{(4)} = 9^{(3)} + 5^{(4)} \end{array}$$

and generally

$$n^{(4)} = n^{(3)} + (n-4)^{(4)}.$$

Proceeding further this way it will be seen that

$$\begin{aligned} n^{(5)} &= n^{(4)} + (n-5)^{(5)}, \\ n^{(6)} &= n^{(5)} + (n-6)^{(6)}, \\ n^{(7)} &= n^{(6)} + (n-7)^{(7)} \\ &\text{etc.} \end{aligned}$$

Therefore, it is concluded that it will be in general

$$n^{(m)} = n^{(m-1)} + (n-m)^{(m)},$$

where it is to be noted, if it was $n < m$, that then the term $(n-m)^{(m)}$ vanishes completely, but if $n = m$, even though $n - m = 0$, that the term $(n-m)^{(m)}$ will nevertheless be $= 1$. Further, if $n - m = 1$, it will also be $(n-m)^{(m)} = 1$. Therefore, it will always be so $0^{(m)} = 1$ as $1^{(m)} = 1$ and $n^{(1)} = 1$.

§33 Having noted these relations among the series $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. they are formed most easily and continued arbitrarily far, which operation will become clear from the table added here:

	1	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}	x^{15}	etc.
$\mathfrak{A} =$	1 +	1 +	1 +	1 +	1 +	1 +	1 +	1 +	1 +	1 +	1 +	1 +	1 +	1 +	1 +	1 +	etc.
			1 +	1 +	2 +	2 +	3 +	3 +	4 +	4 +	5 +	5 +	6 +	6 +	7 +	7 +	etc.
$\mathfrak{B} =$	1 +	1 +	2 +	2 +	3 +	3 +	4 +	4 +	5 +	5 +	6 +	6 +	7 +	7 +	8 +	8 +	etc.
				1 +	1 +	2 +	3 +	4 +	5 +	7 +	8 +	10 +	12 +	14 +	16 +	19 +	etc.
$\mathfrak{C} =$	1 +	1 +	2 +	3 +	4 +	5 +	7 +	8 +	10 +	12 +	14 +	16 +	19 +	21 +	24 +	27 +	etc.
					1 +	1 +	2 +	3 +	5 +	6 +	9 +	11 +	15 +	18 +	23 +	27 +	etc.
$\mathfrak{D} =$	1 +	1 +	2 +	3 +	5 +	6 +	9 +	11 +	15 +	18 +	23 +	27 +	34 +	39 +	47 +	54 +	etc.
						1 +	1 +	2 +	3 +	5 +	7 +	10 +	13 +	18 +	23 +	30 +	etc.
$\mathfrak{E} =$	1 +	1 +	2 +	3 +	5 +	7 +	10 +	13 +	18 +	23 +	30 +	37 +	47 +	57 +	70 +	84 +	etc.
							1 +	1 +	2 +	3 +	5 +	7 +	11 +	14 +	20 +	26 +	etc.
$\mathfrak{F} =$	1 +	1 +	2 +	3 +	5 +	7 +	11 +	14 +	20 +	26 +	35 +	44 +	58 +	71 +	90 +	110 +	etc.
								1 +	1 +	2 +	3 +	5 +	7 +	11 +	15 +	21 +	etc.
$\mathfrak{G} =$	1 +	1 +	2 +	3 +	5 +	7 +	11 +	15 +	21 +	28 +	38 +	49 +	65 +	82 +	105 +	131 +	etc.
									1 +	1 +	2 +	3 +	5 +	7 +	11 +	15 +	etc.
$\mathfrak{H} =$	1 +	1 +	2 +	3 +	5 +	7 +	11 +	15 +	22 +	29 +	40 +	52 +	70 +	89 +	116 +	146 +	etc.
										1 +	1 +	2 +	3 +	5 +	7 +	11 +	etc.
$\mathfrak{I} =$	1 +	1 +	2 +	3 +	5 +	7 +	11 +	15 +	22 +	30 +	41 +	54 +	73 +	94 +	123 +	157 +	etc.
											1 +	1 +	2 +	3 +	5 +	7 +	etc.
$\mathfrak{K} =$	1 +	1 +	2 +	3 +	5 +	7 +	11 +	15 +	22 +	30 +	42 +	55 +	75 +	97 +	128 +	164 +	etc.
												1 +	1 +	2 +	3 +	5 +	etc.
$\mathfrak{L} =$	1 +	1 +	2 +	3 +	5 +	7 +	11 +	15 +	22 +	30 +	42 +	56 +	76 +	99 +	131 +	169 +	etc.
													1 +	1 +	2 +	3 +	etc.
$\mathfrak{M} =$	1 +	1 +	2 +	3 +	5 +	7 +	11 +	15 +	22 +	30 +	42 +	56 +	77 +	100 +	133 +	172 +	etc.
														1 +	1 +	2 +	etc.
$\mathfrak{N} =$	1 +	1 +	2 +	3 +	5 +	7 +	11 +	15 +	22 +	30 +	42 +	56 +	77 +	101 +	134 +	174 +	etc.

etc.

§34 This way the table added here was constructed by addition only and the way of construction is so perspicuous from inspection that a further explanation is not necessary. Therefore, by means of this table the problem to find, *in how many different ways a given number n can be formed from the numbers*

1, 2, 3, $\cdots$ *m by addition*, is solved immediately.
So if it is in question, in how many ways the number 10 can result from these numbers 1, 2 and 3 by addition, it will be $n = 10$ and $m = 3$ and from the table one finds the number of ways to be $= 14$; these ways are

$$\begin{array}{ll}
10 = 1+1+1+1+1+1+1+1+1+1, & 10 = 1+1+1+2+2+3, \\
10 = 1+1+1+1+1+1+1+1+2, & 10 = 1+1+2+2+2+2, \\
10 = 1+1+1+1+1+1+1+3, & 10 = 1+1+2+3+3, \\
10 = 1+1+1+1+1+1+2+2, & 10 = 1+2+2+2+3, \\
10 = 1+1+1+1+1+2+2, & 10 = 1+3+3+3, \\
10 = 1+1+1+1+2+2+2, & 10 = 2+2+2+2+2, \\
10 = 1+1+1+1+3+3, & 10 = 2+2+3+3,
\end{array}$$

If it is in question, in how many ways the number 25 can result from the numbers 1, 2, 3, 4, 5 by addition, having put $n = 25$ and $m = 5$ the number of ways will be found from the table to be $= 377$.
If it is in question, in how many ways the number 50 can result from the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 by addition, having put $n = 50$ and $m = 10$ the number of ways is found to be $= 62740$.
If either the propounded number or the number of parts is greater than those in the table, then it will nevertheless be possible to find the number of cases from the table by means of the formulas found above. For the sake of an example, let it be in question, in how many ways the number 60 can result from the numbers 1, 2, 3, $\cdots$ 20 by addition, it will be $n = 60$ and $m = 20$ and the value of the formula $60^{(20)}$. But $60^{(20)} = 60^{(19)} + 40^{(20)}$, and $60^{(19)} = 60^{(18)} + 41^{(19)}$ and further $60^{(18)} = 60^{(17)} + 42^{(18)}$ and $60^{(17)} = 60^{(16)} + 40^{(17)}$ and so forth. Hence it will finally be

$$60^{(20)} = 40^{(20)} + 41^{(19)} + 42^{(18)} + 43^{(17)} + 44^{(16)} + \cdots + 59^{(1)},$$

which numbers added from the table give 791131; and hence the number 60 can be found from the numbers 1, 2, 3, $\cdots$ m by addition in that many ways.

§35 Further, using this table, both problems propounded by Naude can be solved quickly. First, if it is in question, *in how many different ways a given number N can be split into m different parts*, this, as it was shown above in [§ 19],

will happen in so many ways as the expression $\left(N-\frac{m(m+1)}{2}\right)^{(m)}$, which the table indicates, contains unities.
Therefore, let us demonstrate the application of this table in some examples.

I. *Let it be in question, in how many different ways the number* 25 *can be split into* 5 *different parts.*

Therefore, here it will be $N=25$ and $m=5$, whence $\frac{m(m+1)}{2}=15$, and the answer will contain the formula $10^{(5)}$, which is 30 from the table so that the partition can be done in 30 ways.

II. *Let it be in question, in how many different ways the number* 50 *can be split into* 7 *different parts.*

Here, $N=50$, $m=7$ and $N-\frac{m(m+1)}{2}=22$, whence the number of partitions in question is $22^{(7)}=522$.

III. *Let it be in question, in how many different ways the number* 100 *can be split into* 10 *different parts.*

Since $N=100$ and $m=10$, it will be $N-\frac{m(m+1)}{2}=45$ and the number of partitions will be found to be $45^{(10)}=33401$.

IV. *Let it be in question, in how many different ways the number* 256 *can be split into* 20 *different parts.*

Because of $N=256$ and $m=20$ it will be $N-\frac{m(m+1)}{2}=46$ and the number of partitions will be $46^{(20)}=96271$.

V. *Let it be in question, in how many different ways the number* 270 *can be split into* 20 *different parts.*
Because of $N=270$ and $m=20$ it will be $N-\frac{m(m+1)}{2}=60$ and hence the number of partitions in question is $60^{(20)}$, whose value we found before to be $=791131$. Therefore, the number 270 can be split into 20 different parts in that many different ways.

§36 In like manner, the other problem, *in how many different ways the number N can be split into m not necessarily different parts,* will be solved from the table. For, above [§ 27] we showed that the number of partitions is contained in this formula $(N-m)^{(m)}$, which value can be taken from the table. To understand this solution more easily, let us add some examples.

I. *Let it be in question, in how many different ways the number* 25 *can be split into* 5 *either equal or different parts.*

Here we have $N=25$ and $m=5$, whence $N-m=20$, and the number of partitions will be $20^{(5)}=192$.

II. *Let it be in question, in how many different ways the number* 50 *can be split into* 7 *either equal or different parts.*

Because of $N=50$ and $m=7$ it will be $N-m=43$ and the number of partitions in question will be $43^{(7)}=8946$.

III. *Let it be in question, in how many different ways the number* 50 *can be split into* 10 *either equal or different parts.*

Because of $N=50$ and $m=10$ it will be $N-m=40$ and the number of partitions will be $40^{(10)}=16928$.

IV. *Let it be in question, in how many different ways the number* 60 *can be split into* 12 *either equal or different parts.*

Since $N=60$ and $m=12$, it will be $N-m=48$ and the number of partitions in question will be $48^{(12)}=74287$.

V. *Let it be in question, in how many different ways the number* 80 *can be split into* 20 *either equal or different parts.*

Therefore, it will be $N=80$ and $m=20$, whence $N-m=60$, and the number of partitions will be $60^{(20)}=791131$.

§37 In the horizontal series the table exhibits it is remarkable that the initial terms of these series are identical; and the more initial terms are identical the larger the number m was; so the fifteenth series has fifteen initial terms with all following series in common. Hence one will be able to find a series which corresponds to an infinite number m and which will hence contain the values of this formula $n^{(\infty)}$, which denotes, in how many different ways the number n can be formed from all integer numbers by addition. Therefore, this question deserves some more attention. Since $n^{(\infty)}$ contains completely all partitions of the number n for an arbitrary number of parts, $n^{(\infty)}$ will be the aggregate of the numbers of partitions into 1, 2, 3, 4, $\cdots$ up to n either equal or different parts, since the number n cannot be split into more than n parts. Therefore, it will be

$$n^{(\infty)} = (n-1)^{(1)} + (n-2)^{(2)} + (n-3)^{(3)} + (n-4)^{(4)} + (n-5)^{(5)} + \cdots + (n-n)^{(n)},$$

in which series so the first term $(n-1)^{(1)}$, which denotes the partition into 1 part, as the last $(n-n)^{(n)}$, which denotes the partition into n parts, is 1. Therefore, hence the series of numbers $n^{(\infty)}$, which is exhibited in the last line of the table[1], can be exhibited by addition of the terms from the above series. So it will be

$$6^{(\infty)} = 5^{(1)} + 4^{(2)} + 3^{(3)} + 2^{(4)} + 1^{(5)} + 0^{(6)} = 1 + 3 + 3 + 2 + 1 + 1 = 11,$$

which number is found in the last series under the number 6.

§38 But this operation can be contracted by means of the lemma found above [§ 32], $n^{(m)} = n^{(m-1)} + (n-m)^{(m)}$, whence $n^{(m)} - n^{(m-1)} = (n-m)^{(m)}$. For, since

$$n^{(\infty)} = (n-1)^{(1)} + (n-2)^{(2)} + (n-3)^{(3)} + (n-4)^{(4)} + (n-5)^{(5)} + (n-6)^{(6)} + \text{etc.},$$

if one writes $n-1$ instead of n everywhere, it will be

$$(n-1)^{(\infty)} = (n-1)^{(0)} + (n-2)^{(1)} + (n-3)^{(2)} + (n-4)^{(3)} + (n-5)^{(4)} + (n-6)^{(5)} + \text{etc.},$$

[1] At the end of the paper Euler gives a table of the partitions of all numbers up to 59. This is not reproduced in this translation.

where the term $(n-1)^{(0)}$, whose value is $=0$, is added for the sake of uniformity. Therefore, if the lower series is subtracted from the upper series, by the lemma it will result

$$n^{(\infty)} - (n-1)^{(\infty)}$$
$$= (n-2)^{(1)} + (n-4)^{(2)} + (n-6)^{(3)} + (n-8)^{(4)} + (n-10)^{(5)} + (n-12)^{(6)} + \text{etc.}$$

and so each term $n^{(\infty)}$) is found by means of the preceding $(n-1)^{(\infty)}$ by addition of twice as less terms as before. Therefore, for the sake of an example, it will be

$$12^{(\infty)} = 11^{(\infty)} + 10^{(1)} + 8^{(2)} + 6^{(3)} + 4^{(4)} + 2^{(5)} + 0^{(6)}$$

or

$$12^{(\infty)} = 56 + 1 + 5 + 7 + 5 + 2 + 1 = 77,$$

which number is also found for the value of $12^{(\infty)}$ in the table.

§39 In like manner, this operation can be contracted even further; for, since

$$n^{(\infty)} - (n-1)^{(\infty)} = (n-2)^{(1)} + (n-4)^{(2)} + (n-6)^{(3)} + (n-8)^{(4)} + (n-10)^{(5)} + \text{etc.},$$

if we put $n-2$ instead of n, we will have

$$(n-2)^{(\infty)} - (n-3)^{(\infty)}$$
$$= (n-2)^{(0)} + (n-4)^{(1)} + (n-6)^{(2)} + (n-8)^{(3)} + (n-10)^{(4)} + \text{etc.},$$

where we added the term $(n-2)^{(0)} = 0$ for the sake of uniformity. Now, by subtracting this series from the above one by means of the lemma we will obtain

$$(n)^{(\infty)} - (n-1)^{(\infty)} - (n-2)^{(\infty)} + (n-3)^{(\infty)}$$
$$= (n-3)^{(1)} + (n-6)^{(2)} + (n-9)^{(3)} + (n-12)^{(4)} + (n-15)^{(5)} + \text{etc.}$$

Therefore, if this series is called P, it will be

$$n^{(\infty)} = (n-1)^{(\infty)} + (n-2)^{(\infty)} - (n-3)^{(\infty)} + P.$$

Therefore, in order to define the term $n^{(\infty)}$ in the series in question, except for the value of P one has to know three preceding terms. Proceeding this way the quantity P will finally vanish and each term of this series will be defined only by the preceding terms, which is the property of recurring series.

§40 That this series is indeed recurring is also obvious from its origin, since it results from the expansion of this fraction

$$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)\text{etc.}}$$

Therefore, the relation scale of this series will be found, if this denominator is actually expanded by multiplication. But after this multiplication the denominator will be found expressed in the following way:

$$1-x-x^2+x^5+x^7-x^{12}-x^{15}+x^{22}+x^{26}-x^{35}-x^{40}+x^{51}+x^{57}-x^{70}-x^{77}+\text{etc.}$$

It seems hard to discover, by what kind of formula the exponents of the powers of x are expressed; but nevertheless it becomes clear very soon that the signs of each two terms alternate. Furthermore, the exponents are all seen to follow a certain law, whence the general one is concluded to be $x^{n(3n\pm 1):2}$. Of course, no other powers than those occur, whose exponents are contained in the formula $\frac{3nn\pm n}{2}$ in such a way that the powers resulting from odd numbers assumed for n have the sign $-$, but the powers formed from even numbers have the sign $+$.

§41 Therefore, this formula gives us the relation scale of the series in question; for, now we have

$$n^{(\infty)}=(n-1)^{(\infty)}+(n-2)^{(\infty)}-(n-5)^{(\infty)}-(n-7)^{(\infty)}+(n-12)^{(\infty)}+(n-15)^{(\infty)}$$
$$-(n-22)^{(\infty)}-(n-26)^{(\infty)}+(n-35)^{(\infty)}+(n-40)^{(\infty)}-(n-51)^{(\infty)}-(n-57)^{(\infty)}+\text{etc.}$$

This rule for the progression will be seen to be true by testing it. For, let $n=30$; one will find

$$30^{(\infty)}=29^{(\infty)}+28^{(\infty)}-25^{(\infty)}-23^{(\infty)}+18^{(\infty)}+15^{(\infty)}-8^{(\infty)}-4^{(\infty)};$$

for, having taken these numbers from the table

$$5604 = 4565 + 3718 - 1958 - 1255 + 385 + 176 - 22 - 5.$$

And this way the series can be continued arbitrarily far.

§42 But since the series for the value $m = 20$ was already formed, from it the series in question for the value $m = \infty$ can be found a bit more easily. For, since the series $n^{(20)}$ is formed from the expansion of this fraction

$$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)\cdots(1-x^{20})},$$

but the series $n^{(\infty)}$ from the expansion of this fraction

$$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)\cdots(1-x^{\infty})},$$

it is obvious, if this series is multiplied by

$$(1-x^{21})(1-x^{22})(1-x^{23})(1-x^{24})(1-x^{25})\text{etc.}$$

or by

$$\begin{gathered}
1 - x^{21} - x^{22} - x^{23} - x^{24} - x^{25} - x^{26} - x^{27} - \text{etc.} \\
+x^{43} + x^{44} + 2x^{45} + 2x^{46} + 3x^{47} + 3x^{48} + 4x^{49} + 4x^{50} + \text{etc.} \\
-x^{66} - x^{67} - 2x^{68} - 3x^{69} - 4x^{70} - 5x^{71} - 6x^{72} - 8x^{73} - 10x^{74} - \text{etc.} \\
+x^{90} + x^{91} + 2x^{92} + 3x^{93} + 5x^{94} + 6x^{95} + 9x^{96} + 11x^{97} + 15x^{98} + \text{etc.} \\
-x^{115} - x^{116} - 2x^{117} - 3x^{118} - 5x^{119} - 7x^{120} - 10x^{121} - 13x^{122} - 18x^{123} - \text{etc.} \\
\text{etc.},
\end{gathered}$$

that then the first must result. Hence it is concluded that it will be

$$\begin{aligned}
n^{(20)} &= n^{(\infty)} - (n-21)^{(\infty)} - (n-22)^{(\infty)} - (n-23)^{(\infty)} - (n-24)^{(\infty)} - \text{etc.} \\
&+ (n-43)^{(\infty)} + (n-44)^{(\infty)} + 2(n-45)^{(\infty)} + 2(n-46)^{(\infty)} + 3(n-47)^{(\infty)} + \text{etc.} \\
&- (n-66)^{(\infty)} - (n-67)^{(\infty)} - 2(n-68)^{(\infty)} - 3(n-69)^{(\infty)} - 4(n-70)^{(\infty)} - \text{etc.} \\
&+ (n-90)^{(\infty)} + (n-91)^{(\infty)} + 2(n-92)^{(\infty)} + 3(n-93)^{(\infty)} + 5(n-94)^{(\infty)} + \text{etc.} \\
&- (n-115)^{(\infty)} - (n-116)^{(\infty)} - 2(n-117)^{(\infty)} - 3(n-118)^{(\infty)} - 5(n-119)^{(\infty)} - \text{etc.}
\end{aligned}$$

etc.,

the coefficients of which series proceed according to the above series for the partitions numbers into 2, 3, 4, 5, 6 etc. parts.

§43 Let $\int(n-21)^{(\infty)}$ denote the sum of all terms of the series $n^{(\infty)}$, which series is

$$1+1+2+3+5+7+11+15+22+30+\text{etc.}$$

up to the term $(n-21)^{(\infty)}$ inclusively; and in like manner let in general $\int p^{(\infty)}$ be the sum of all terms of the same series up to the term $p^{(\infty)}$ inclusively; since these sums are easily formed successively, it will be

$$\begin{aligned} n^{(20)} = n^{(\infty)} & - \int(n-21)^{(\infty)} + \int(n-43)^{(\infty)} + \int(n-45)^{(\infty)} + \int(n-47)^{(\infty)} \quad + \text{etc.} \\ & - \int(n-66)^{(\infty)} - \int(n-68)^{(\infty)} - \int(n-69)^{(\infty)} - \int(n-70)^{(\infty)} \quad - \text{etc.} \\ & + \int(n-90)^{(\infty)} + \int(n-92)^{(\infty)} + \int(n-93)^{(\infty)} + 2\int(n-94)^{(\infty)} + \text{etc.} \\ & \text{etc.} \end{aligned}$$

And hence it will be

$$\begin{aligned} n^{(\infty)} = n^{(20)} & + \int(n-21)^{(\infty)} - \int(n-43)^{(\infty)} - \int(n-45)^{(\infty)} - \int(n-47)^{(\infty)} \quad - \text{etc.} \\ & + \int(n-66)^{(\infty)} + \int(n-68)^{(\infty)} + \int(n-69)^{(\infty)} + \int(n-70)^{(\infty)} \quad + \text{etc.} \\ & - \int(n-90)^{(\infty)} - \int(n-92)^{(\infty)} - \int(n-93)^{(\infty)} - 2\int(n-94)^{(\infty)} - \text{etc.} \end{aligned}$$

By means of this formula, if the number n is not very large, from the series for the partition into 20 parts the series $n^{(\infty)}$ is easily constituted and this way it is exhibited in the constructed table, since the excesses of the terms $n^{(\infty)}$ over the terms $n^{(20)}$ were assigned everywhere.

§44 Therefore, having constructed this series and then having propounded an arbitrary number one will be able to define, in how many ways it can be split into its parts. So it is plain that the number 10 can result in 42 ways by addition; and the number 59 can result in so many ways by addition as the number 831820 indicates it. But if greater numbers are propounded, then the table exhibited here must be continued for each case or one has to investigate the number by the prescriptions given here. Hence a new problem arises, *in which for a certain propounded number the number of all partitions into different parts is in question*; this problem is solved by this expression

$$(1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6)\text{etc.}$$

For, having multiplied these factors by each other a series results, in which each coefficient will show, in how many different ways the exponent of x can be split into different parts.

§45 But if this product is actually expanded, one will find this series

$$1+x+x^2+2x^3+2x^4+3x^5+4x^6+5x^7+6x^8+8x^9+10x^{10}+12x^{11}$$
$$+15x^{12}+18x^{13}+22x^{14}+27x^{15}+32x^{16}+38x^{17}+46x^{18}+54x^{19}+64x^{20}$$
$$76x^{21}+89x^{22}+\text{etc.};$$

since this is a product of infinitely many factors of such a simple structure, it seems worth one's complete attention. And first, it is certainly obvious that the coefficients of these terms are mostly even and only those are odd which correspond to powers of x, whose exponents are contained in the form $\frac{3nn\pm n}{2}$; the reason for this phenomenon is the same as for that one we saw in the exponent of the same form $\frac{3nn\pm n}{2}$ in the expansion of the product $(1-x)(1-x^2)(1-x^3)(1-x^4)$etc. But since

$$(1+x)(1+x^2)(1+x^3)(1+x^4)\text{etc.}=\frac{(1-x^2)(1-x^4)(1-x^6)(1-x^8)\text{etc.}}{(1-x)(1-x^2)(1-x^3)(1-x^4)\text{etc.}},$$

it is apparent that the series found before is expressed by this fraction

$$\frac{1-x^2-x^4+x^{10}+x^{14}-x^{24}-x^{30}+x^{44}+x^{52}-x^{70}-x^{80}+\text{etc.}}{1-x^1-x^2+x^5+x^7-x^{12}-x^{15}+x^{22}+x^{26}-x^{35}-x^{40}+\text{etc.}},$$

whence it can be formed like recurring series.

§46 But without any doubt this series is most easily constructed from its nature itself, according to which the coefficient of each terms must indicate, in how many different ways the exponent of x can be split into different parts. Let N be the coefficient of the power x^n in this series and it will be

$$N = (n-1)^{(1)} + (n-3)^{(2)} + (n-6)^{(3)} + (n-10)^{(4)} + (n-15)^{(5)} + (n-21)^{(6)} + \text{etc.};$$

for, $(n-1)^{(1)} = 1$ indicates that the number n can consist of one part in one single way, $(n-3)^{(2)}$ shows, in how many ways the number n can be split into two different parts, $(n-6)^{(3)}$ shows, in how many ways the number n can be split into three different parts, and so forth; hence also this series can be continued arbitrarily far by means of the tables. Furthermore, it is remarkable, if the number of partitions into an even number of parts is taken negatively, that this resulting expression

$$(n-1)^{(1)} - (n-3)^{(2)} + (n-6)^{(3)} - (n-10)^{(4)} + (n-15)^{(5)} - (n-21)^{(6)} + \text{etc.}$$

is always $= 0$, if n was not a number contained in the form $\frac{3zz \pm z}{2}$, but if n is contained in this form, that the value of that expression is either $+1$ or -1, depending on whether z was an odd or even number.

§47 As up to this point we allowed all integer numbers to constitute the parts, so, by restricting the properties of the parts, the number of questions could be increased to infinity; but will we not spend more time on this task, since a certain method to solve questions of this kind was already given. It shall suffice, to have noted an extraordinary property of the partitions into odd parts. Since

$$(1+x)(1+x^2)(1+x^3)(1+x^4)\text{etc.} = \frac{1}{(1-x)(1-x^3)(1-x^5)(1-x^7)(1-x^9)\text{etc.}},$$

which formula follows from the equation exhibited in § 45, hence it follows that each number can be produced from only odd numbers by addition in so many ways as the same number can be split into different parts. So, since the number 10 can be split into different parts in ten ways, which are

$$
\begin{array}{ll}
10 = 10, & 10 = 1 + 2 + 7, \\
10 = 1 + 9, & 10 = 1 + 3 + 6, \\
10 = 2 + 8, & 10 = 1 + 4 + 5, \\
10 = 3 + 7, & 10 = 2 + 3 + 5, \\
10 = 4 + 6, & 10 = 1 + 2 + 3 + 4,
\end{array}
$$

the same number 10 can also be produced in ten ways from only odd numbers by addition as follows

$$
\begin{array}{ll}
10 = 1+1+1+1+1+1+1+1+1+1, & 10 = 1 + 3 + 3 + 3, \\
10 = 1+1+1+1+1+1+1+3, & 10 = 1 + 1 + 1 + 1 + 3 + 3, \\
10 = 1+1+1+1+1+5, & 10 = 1 + 1 + 3 + 5, \\
10 = 1+1+1+7, & 10 = 3 + 7, \\
10 = 1+9, & 10 = 5 + 5.
\end{array}
$$

§48 But having put these considerations aside, I proceed to investigate, how each number can be formed from the terms of the geometric progression 1, 2, 4, 8, 16, 32 etc. by addition. And first, if these parts must all be different, the question will be resolved by expansion of this expression

$$
s = (1 + x)(1 + x^2)(1 + x^4)(1 + x^8)(1 + x^{16})(1 + x^{32})\text{etc.}
$$

For, having actually done the multiplication, the coefficient of each term will indicate, in how many ways the exponents of the corresponding power of x can be produced from the numbers of the geometric progression 1, 2, 4, 8, 16 etc. by addition. Therefore, since each number was observed that it can be resolved in one way like this, it is to be demonstrated that in this series all powers of x occur and they all have the same coefficient, which is $= 1$.

§49 To demonstrate this, let us put that

$$
s = 1 + \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \zeta x^6 + \eta x^7 + \theta x^8 + \text{etc.}
$$

and, in order to find the values of the coefficients α, β, γ, δ, let us put xx instead of x and let the value resulting for s this way be $= t$; it will be

$$t = (1+x^2)((1+x^4)(1+x^8)(1+x^{16})(1+x^{32})\text{etc.}$$

and hence it will be $s = (1+x)t$. Having considered this relation in the series because of

$$t = 1 + \alpha x^2 + \beta x^4 + \gamma x^6 + \delta x^8 + \varepsilon x^{10} + \text{etc.}$$

one will have

$$(1+x)t = 1 + x + \alpha x^2 + \alpha x^3 + \beta x^4 + \beta x^5 + \gamma x^6 + \gamma x^7 + \delta x^8 + \delta x^9 + \text{etc.};$$

since this series must be equal to the series s, the comparison of the coefficients will give

$$\begin{array}{llll} \alpha = 1, & \delta = \beta, & \eta = \gamma, & \varkappa = \varepsilon, \\ \beta = \alpha, & \varepsilon = \beta, & \theta = \delta, & \lambda = \varepsilon, \\ \gamma = \alpha, & \zeta = \gamma, & \iota = \delta, & \mu = \zeta \\ & \text{etc.} & & \end{array}$$

whence it is obvious that the single coefficients are all equal to 1 and therefore

$$s = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + \text{etc.} = \frac{1}{1-x};$$

the same is perspicuous per se, since

$$(1-x)(1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})\text{etc.} = 1.$$

§50 But if it is in question, in how many different ways each number can be formed from not necessarily different terms of the geometric progression 1, 2, 4, 8, 16 etc. by addition, the solution is to be derived from the expansion of this fraction

$$s = \frac{1}{(1-x)(1-x^2)(1-x^4)(1-x^8)(1-x^{16})(1-x^{32})\text{etc.}};$$

for, in this series, if expanded, the coefficient of each term will show, in how many different ways the exponent of the corresponding power of x can result from the terms of the propounded geometric progression by addition. Let us put xx instead of x and let the value of s go over into t; it will be

$$t = \frac{1}{(1-x^2)(1-x^4)(1-x^8)(1-x^{16})\text{etc.}} = (1-x)s;$$

therefore, let

$$s = 1 + \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \zeta x^6 + \eta x^7 + \theta x^8 + \iota x^9 + \text{etc.};$$

it will be

$$\begin{array}{rcccccccccccc} (1-x)s = & 1 & + \alpha x & + \beta x^2 & + \gamma x^3 & + \delta x^4 & + \varepsilon x^5 & + \zeta x^6 & + \eta x^7 & + \theta x^8 & + \iota x^9 & + \text{etc.} \\ & & -1 & -\alpha & -\beta & -\gamma & -\delta & -\varepsilon & -\zeta & -\eta & -\theta & -\text{etc.} \\ = t = & 1 & & + \alpha x^2 & & + \beta x^4 & & + \gamma x^6 & & + \delta x^8 & & + \text{etc.}, \end{array}$$

whence from the equality of the homogeneous terms one will obtain

$$\begin{array}{lll} \alpha = 1 \qquad\quad = 1, & \eta = \zeta \qquad\quad = 6, & \nu = \mu \qquad\quad = 20, \\ \beta = \alpha + \alpha = 2, & \theta = \eta + \delta = 10, & \xi = \nu + \eta = 26, \\ \gamma = \beta \qquad\quad = 2, & \iota = \theta \qquad\quad = 10, & o = \xi \qquad\quad = 26, \\ \delta = \gamma + \beta = 4, & \varkappa = \iota + \varepsilon = 14, & \pi = o + \theta = 36, \\ \varepsilon = \delta \qquad\quad = 4, & \lambda = \varkappa \qquad\quad = 14, & \rho = \pi \qquad\quad = 36, \\ \zeta = \varepsilon + \gamma = 6, & \mu = \lambda + \zeta = 20, & \sigma = \rho + \iota = 46, \\ & \text{etc.} & \end{array}$$

§51 This series is remarkable, both since each two terms are equal and since it easily continued arbitrarily far. Having continued it further, it will look like this:

$$1 + x + 2x^2 + 2x^3 + 4x^4 + 4x^5 + 6x^6 + 6x^7 + 10x^8 + 10x^9 + 14x^{10} + 14x^{11}$$
$$+20x^{12} + 20x^{13} + 26x^{14} + 26x^{15} + 36x^{16} + 36x^{17} + 46x^{18} + 46x^{19} + 60x^{20} + 60x^{21}$$
$$+74x^{22} + 74x^{23} + 94x^{24} + 94x^{25} + 114x^{26} + 114x^{27} + 140x^{28} + 140x^{29} + 166x^{30}$$

$$+166x^{31}+202x^{32}+202x^{33}+238x^{34}+238x^{35}+284x^{36}+284x^{37}+\text{etc.}$$

Therefore, from this series it is plain that the number 30 can result in one hundred and sixty-six ways from the geometric progression by addition, for example. Furthermore, the attentive reader will easily see that the law of this progression cannot by expressed by a general term by any means, although this series is indeed recurring; but its relation scale is extended to infinity. But this infinite product

$$(1-x)(1-x^2)(1-x^4)(1-x^8)(1-x^{16})(1-x^{32})\text{etc.},$$

if expanded, will give the scale of relation. To find it, put this product $= p$, which goes over into q, if one puts x^2 instead of x, and it will be

$$q=(1-x^2)(1-x^4)(1-x^8)(1-x^{16})\text{etc.}=\frac{p}{1-x}$$

or $p=(1-x)q$. Therefore, set

$$p=1+\alpha x+\beta x^2+\gamma x^3+\delta x^4+\varepsilon x^5+\zeta x^6+\eta x^7+\theta x^8+\iota x^9+\varkappa x^{10}+\text{etc.},$$

and it will be

$$(1-x)q=1-x+\alpha x^2-\alpha x^3+\beta x^4-\beta x^5+\gamma x^6-\gamma x^7+\delta x^8-\delta x^9+\varepsilon x^{10}-\text{etc.},$$

whence by equating the coefficients one obtains

$$\begin{array}{lll}
\alpha=-1=-1, & \theta=\delta=-1, & o=-\eta=+1,\\
\beta=\alpha=-1, & \iota=-\delta=+1, & \pi=\theta=-1,\\
\gamma=-\alpha=+1, & \varkappa=\varepsilon=+1, & \rho=-\theta=+1,\\
\delta=\beta=-1, & \lambda=-\delta=-1, & \sigma=\iota=+1,\\
\varepsilon=-\beta=+1, & \mu=\zeta=+1, & \tau=-\iota=-1,\\
\zeta=\gamma=+1, & \nu=-\zeta=-1, & \upsilon=\varkappa=+1,\\
\eta=-\gamma=-1, & \xi=\eta=-1, & \varphi=-\varkappa=-1
\end{array}$$

etc.

§52 Therefore, the coefficients of the series p, which results from the expansion of this product

$$(1-x)(1-x^2)(1-x^4)(1-x^8)(1-x^{16})(1-x^{32})\text{etc.},$$

are all either $+1$ or -1 and nevertheless do not follow a law assignable by customary methods; for, it will be

$$p = 1 - x^1 - x^2 + x^3 - x^4 + x^5 + x^6 - x^7 - x^8 + x^9 + x^{10} - x^{11} + x^{12} - x^{13} - x^{14}$$
$$+ x^{15} - x^{16} + x^{17} + x^{18} - x^{19} + x^{20} - x^{21} - x^{22} + x^{23} + x^{24} - x^{25} - x^{26} + x^{27} - x^{28}$$
$$+ x^{29} + x^{30} - x^{31} - x^{32} + x^{33} + x^{34} - x^{35} + x^{36} - x^{37} - x^{38} + x^{39} + x^{40} - x^{41} - x^{42}$$
$$+ x^{43} - x^{44} + \text{etc.},$$

where it is to be noted that each power with an odd exponent x^{2n+1} has the contrary sign to that the power x^{2n}, and the sign of this always agrees with the sign of the power x^n; hence the sign of each power is easily assigned. If, for the sake of an example, the sign of the power x^{1745} is in question, just considering the signs it will be

$$x^{1745} = -x^{1744} = -x^{872} = -x^{436} = -x^{218} = -x^{109} = +x^{108} = +x^{54} = +x^{27}$$
$$= -x^{26} = -x^{13} = +x^{12} = +x^6 = +x^3 = -x^2 = -x^1;$$

therefore, the sign of the power x^{1745} is contrary to the sign of the power x^1; since this sign is $-$, that one will be $+$.